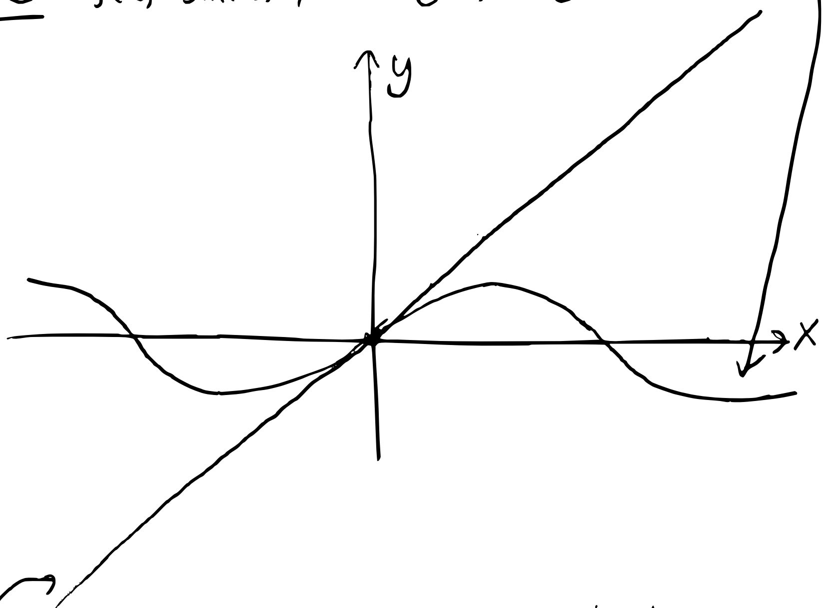
Tangent planes and approximations

Recall that, in Calculus I, the tangent lines can be used to approximate the functions.

Example $f(x)=\sin(x)$, the graph $y=f(x)$ is this.



This is the tangent line at $x=0$, which is

$$\boxed{y = f'(0)(x - f(0)) = 1\cdot(x-0) = x}$$

For x close to 0, $y=\sin(x)$ is close to $y=x$.

Thus, for x close to 0, $\sin(x)$ is close to x.

$\sin(0.1) = 0.09983\cdots \sim 0.1$

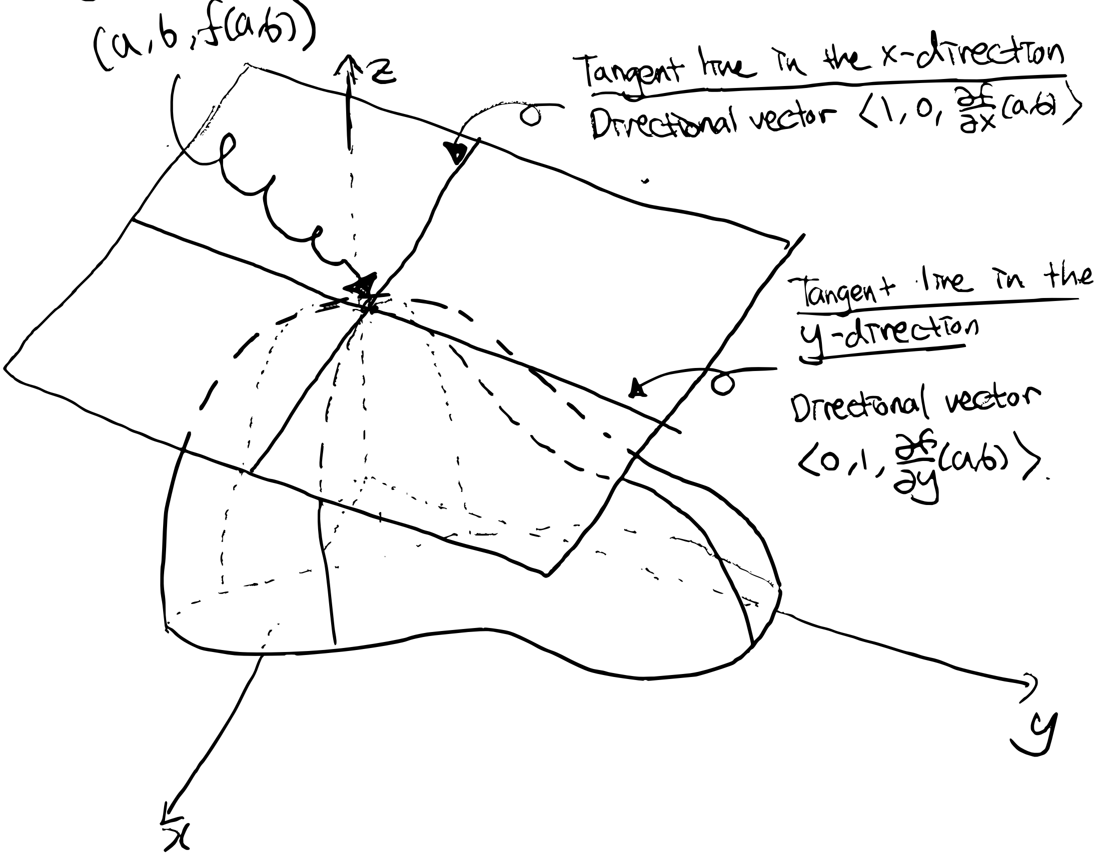
$\sin(0.15) = 0.14943\cdots \sim 0.15$

$\sin(0.2) = 0.19866\cdots \sim 0.2$

⚠ The approximation doesn't work if x is far from 0.

Similarly, we can approximate functions of several variables using the notion of tangency.

For a function $f(x,y)$ with two variables, the graph is tangent to a tangent plane.



This is the plane that contains both tangent lines we learned from partial derivatives.

along x: $\langle a, b, f(a,b) \rangle + t \langle 1, 0, \frac{\partial f}{\partial x}(a,b) \rangle$

along y: $\langle a, b, f(a,b) \rangle + t \langle 0, 1, \frac{\partial f}{\partial y}(a,b) \rangle$

The tangent plane is therefore

- passing through $(a, b, f(a,b))$
- parallel to $\langle 1, 0, f_x(a,b)\rangle$, $\langle 0, 1, f_y(a,b)\rangle$.

$$= \text{normal to } \langle 1, 0, f_x(a,b)\rangle \times \langle 0, 1, f_y(a,b)\rangle$$

$$= \langle -f_x(a,b), -f_y(a,b), 1\rangle$$

Therefore, the tangent plane has an equation

$$-f_x(a,b)(x-a) - f_y(a,b)(y-b) + 1\cdot(z - f(a,b)) = 0,$$

or

$$\boxed{z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).}$$

Example Find the tangent plane to the graph

$z = e^{xy}\sin y$ at the point $(2, 0, 0)$.

Solution For $f(x,y) = e^{xy}\sin y$,

$$f_x(x,y) = y e^{xy}\sin y, \quad f_y(x,y) = x e^{xy}\sin y + e^{xy}\cos y$$

$$\Rightarrow \quad f_x(2,0) = 0, \quad f_y(2,0) = 0 + 1 = 1.$$

So the tangent plane has equation

$$z = f(2,0) + f_x(2,0)(x-2) + f_y(2,0)(y-0)$$

$$= 0 + 0\cdot(x-2) + 1\cdot(y-0) = y.$$

$$\boxed{z = y.}$$

If the function z is implicitly defined, then the partial derivatives are $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, and the tangent plane at (a,b,c) is

$$\boxed{z = c + \frac{\partial z}{\partial x}(a,b)\,(x-a) + \frac{\partial z}{\partial y}(a,b)\,(y-b)}.$$

Example Find the tangent plane of $x^2+y^2-z^2=1$ at $(1,1,1)$.

Solution Taking $\frac{\partial}{\partial x}$, we get

$$2x - 2z\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{x}{z} = \frac{\partial z}{\partial x}$$

Taking $\frac{\partial}{\partial y}$, we get

$$2y - 2z\frac{\partial z}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial z}{\partial y} = \frac{y}{z}.$$

So $\frac{\partial z}{\partial x}(1,1) = 1 \qquad \frac{\partial z}{\partial y}(1,1) = 1$

Tangent plane has equation

$$\Rightarrow \quad z = 1 + 1\cdot(x-1) + 1\cdot(y-1)$$

$$= x + y + 1$$

$$\Rightarrow \quad \boxed{z = x + y + 1}$$

Error approximation

You can approximate the function $f(x,y)$ using the tangent plane. Namely, if (x,y) is close to (a,b),

$$f(x,y) \sim f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

This means that

$$f(a+0.01,\, b-0.03)$$
$$\sim f(a,b) + f_x(a,b)\cdot 0.01 + f_y(a,b)\cdot(-0.03)$$

This gives us a way to approximate the error of f in terms of the error of x and y.

One writes Δx for the error of x. (Sometimes called the differential)

Δy ... y

Δf ... f

From the tangent plane approximation, we get

$$\boxed{\Delta f \sim f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y}$$

or $$\boxed{\Delta f \sim \frac{\partial f}{\partial x}(a,b)\,\Delta x + \frac{\partial f}{\partial y}(a,b)\,\Delta y}$$

Example You bought a baseball launcher that launches a baseball at a speed of 20 m/s at an angle of $\frac{\pi}{6}$ ($=30°$) from the ground.

Neglecting the effect of air resistance, the maximum height of a projectile launched on the ground at the initial speed of v (m/s) and at the initial angle of θ (radians) from the ground is

$$h = \frac{v^2 \sin^2\theta}{20} \quad (m)$$

According to the specs of the baseball launcher, the maximum height a baseball can reach (when launched from the ground) is $\frac{20^2 \cdot \sin^2\frac{\pi}{6}}{20} = 20 \times \left(\frac{1}{2}\right)^2 = 5$ (m).

The indoor soccer field around your house is very large in terms of the area, and has a ceiling at 5.5m high. So if you launch a baseball in the indoor soccer field on the ground, the baseball will not hit the ceiling, according to the specs.

But in fact, you find ~~out~~ that the baseball launcher launches at the speed of 20 ± 0.5 (m/s) and at the angle of $\frac{\pi}{6} \pm 0.01$ (radians).

Q Is there a chance that a launched baseball may hit the ceiling?

<u>Solution</u> We have $h(v,\theta) = \frac{v^2 \sin^2\theta}{20}$, so

$$\Delta h \sim h_v(20, \tfrac{\pi}{6})\,\Delta v + h_\theta(20, \tfrac{\pi}{6})\,\Delta\theta.$$

Note $h_v(v,\theta) = \frac{v\sin^2\theta}{10} \rightsquigarrow h_v(20, \frac{\pi}{6}) = \frac{20\sin^2(\frac{\pi}{6})}{10} = \frac{20 \times \frac{1}{4}}{10} = \frac{1}{2}$,

$h_\theta(v,\theta) = \frac{2v^2\sin\theta\cos\theta}{20} \rightsquigarrow h_\theta(v,\theta) = \frac{2\times 20^2 \times \sin(\frac{\pi}{6})\cos(\frac{\pi}{6})}{20}$

$= 40 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} = 10\sqrt{3}.$

$\Rightarrow \Delta h \sim \frac{1}{2}\Delta v + 10\sqrt{3}\,\Delta\theta.$ (with $-0.5 \leq \Delta v \leq 0.5$, $-0.01 \leq \Delta\theta \leq 0.01$.)

To make sure that the ball does not hit the ceiling, you would want $h < 5.5$, or $\Delta h < 0.5$ (since $h(20, \frac{\pi}{6}) = 5$).

Δh is maximized if Δv, $\Delta\theta$ are maximized.

So the maximum possible value of Δh is $\sim$

$$\frac{1}{2}\cdot 0.5 + 10\sqrt{3}\cdot 0.01 = \frac{1}{4} + \frac{\sqrt{3}}{10} = \frac{5+2\sqrt{3}}{20}.$$

Since $2\sqrt{3} = \sqrt{12} < \sqrt{16} = 4$, $\frac{5+2\sqrt{3}}{20} < \frac{5+4}{20} = \frac{9}{20} < 0.5.$

$\Rightarrow$ Can conclude that, even under the inaccuracies of the specs, a launched baseball CANNOT hit the ceiling.

What we showed is that, if v is within 20 ± 0.5,
and θ is within $\frac{\pi}{6} \pm 0.01$,

we have that h is within $5 \pm \left(\frac{5+2\sqrt{3}}{20}\right) \sim 5 \pm 0.45$.

We call these numbers for the possible amount of error the margin of error (MOE in short).

From the error calculation, for $f(x,y)$ at $(x,y)=(a,b)$, we get

$$MOE_f = \left|\frac{\partial f}{\partial x}(a,b)\right| MOE_x + \left|\frac{\partial f}{\partial y}(a,b)\right| MOE_y$$

where MOE_f, MOE_x, MOE_y are the margins of errors of f, x, y.

In other words, if x is within $a \pm MOE_x$ and if y is within $b \pm MOE_y$, then $f(x,y)$ is within $f(a,b) \pm MOE_f$, where $MOE_f = \left|\frac{\partial f}{\partial x}(a,b)\right| MOE_x + \left|\frac{\partial f}{\partial y}(a,b)\right| MOE_y$.

Similarly, for a three-variable function $f(x,y,z)$, the approximation can be given by

$$f(x,y,z) \sim f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$$

Or, in terms of the error/differential,

$$df = f_x\,dx + f_y\,dy + f_z\,dz,$$

$$\text{or } df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

In terms of MOE,

$$MOE_f = \left|\frac{\partial f}{\partial x}(a,b,c)\right| MOE_x + \left|\frac{\partial f}{\partial y}(a,b,c)\right| MOE_y + \left|\frac{\partial f}{\partial z}(a,b,c)\right| MOE_z.$$

Example Suppose you have a water tank. You measured its sides and you got

$$(2 \pm 0.05)\,m \times (1.5 \pm 0.03)\,m \times (3 \pm 0.07)\,m.$$

What is the expected volume, and how much is the margin of error?

Solution The expected volume is just $2 \times 1.5 \times 3 = 9\ m^3$.

The volume of the tank with three sides x, y, z (in meters) is given by $V(x,y,z) = xyz$. So

$$dV = V_x dx + V_y dy + V_z dz = yz\,dx + xz\,dy + xy\,dz$$

$$= 1.5 \times 3\,dx + 2 \times 3\,dy + 2 \times 1.5\,dz$$

$$= 4.5\,dx + 6\,dy + 3\,dz$$

In our case, we are interested in MOE_V.

$$MOE_V = 4.5\,MOE_x + 6\,MOE_y + 3\,MOE_z$$

$$MOE_x = 0.05,\ MOE_y = 0.03,\ MOE_z = 0.07$$

$$\Rightarrow MOE_V = 0.05 \times 4.5 + 0.03 \times 6 + 0.07 \times 3$$

$$= 0.225 + 0.18 + 0.21 = 0.615$$

So $V = (9 \pm 0.615)\,m^3$.

More approximation

From the previous discussion, we learned that we can approximate if we know the partial derivatives.

If we could approximate even these partial derivatives, we would be able to approximate the function without knowing about the function.

The basic idea comes from that

$$f_x(x,y) = \lim_{e \to 0} \frac{f(x+e,y) - f(x,y)}{e}$$

$$f_y(x,y) = \lim_{e \to 0} \frac{f(x,y+e) - f(x,y)}{e},$$ so you can approximate

$$f_x(x,y) \sim \frac{f(x+e,y) - f(x,y)}{e}, \quad f_y(x,y) \sim \frac{f(x,y+e) - f(x,y)}{e}$$

if e is relatively small.

A. Approximation using the table

Example We have the table of perceived temperature $I(T,H)$ given the actual temperature T and the humidity H.

Humidity (%)

Actual Temp (°F)

T \ H	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	135	135	141	147	154	161	168

Approximate the value of $I(96.5, 71)$.

Solution We have

$$I(96.5, 71) \sim I(96, 70) + I_T(96, 70) \cdot 0.5 + I_H(96, 70) \cdot 1$$
$$= 125 + I_T(96, 70) \cdot 0.5 + I_H(96, 70)$$

$$I_T(96, 70) \sim \frac{I(98, 70) - I(96, 70)}{2} = \frac{133 - 125}{2} = 4$$

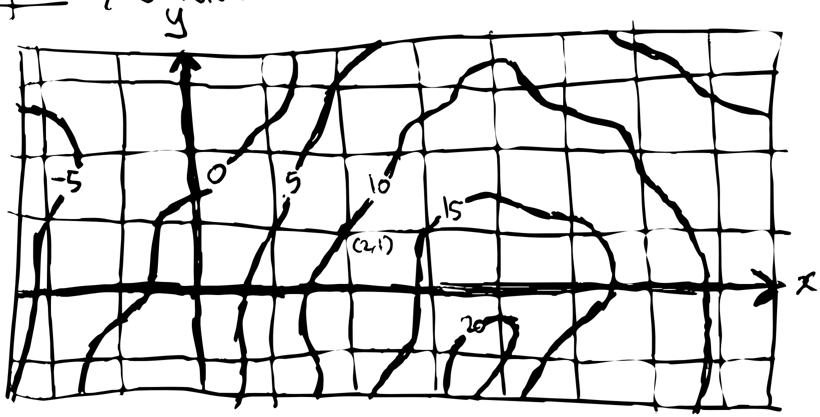
$$I_H(96, 70) \sim \frac{I(96, 75) - I(96, 70)}{5} = \frac{130 - 125}{5} = 1,$$

So $I(96.5, 71) \sim 125 + 4 \times 0.5 + 1 = 128$.

B. Approximation using the contour map

Again, this is about approximating the partial derivatives.

Example You have the contour map for $f(x,y)$.



Approximate the value of $f(2.1, 0.8)$.

Solution

$$f(2.1, 0.8) \sim f(2,1) + f_x(2,1)\cdot 0.1 + f_y(2,1)\cdot(-0.2)$$

$$= 10 + 0.1\cdot f_x(2,1) - 0.2\cdot f_y(2,1)$$

Since $f(3,1) = 15$, $f_x(2,1) \sim \dfrac{f(3,1)-f(2,1)}{1} = \dfrac{15-10}{1} = 5$.

Since $f(2,3) = 5$, $f_y(2,1) \sim \dfrac{f(2,3)-f(2,1)}{2} = \dfrac{5-10}{2} = -2.5$

So $f(2.1, 0.8) \sim 10 + 0.5 + 0.5 = 11$.